

# Viscoplasticity: A Thermodynamic Formulation

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# VISCOPLASTICITY: A THERMODYNAMIC FORMULATION†

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## ABSTRACT

A thermodynamic foundation using the concept of internal state variables is given for a general theory of viscoplasticity, as it applies to initially isotropic materials. Three fundamental internal state variables are admitted; they are: a tensor-valued back stress for kinematic effects, and the scalar-valued drag and yield strengths for isotropic effects. All three are considered to phenomenologically evolve according to competitive processes between strain hardening, strain-induced dynamic recovery, and time-induced static recovery. Within this phenomenological framework, a thermodynamically admissible set of evolution equations is put forth. This theory allows each of the three fundamental internal variables to be composed as a sum of independently evolving constituents.

## 1. INTRODUCTION

The theoretical development of viscoplasticity has its origin with the works of BINGHAM & GREEN [1], HOHENEMSER & PRAGER [2], OLDROYD [3], ODQVIST [4], STOWELL [5], and PRAGER [6], whose models do not contain evolving internal state variables. The field started to gain inertia in the mid 1960's when internal state variable models began to appear in the theories of PERZYNA [7] and ARMSTRONG & FREDERICK [8]. With the advent of the computer, rapid advances were made in the 1970's with the modeling efforts of BODNER & PARTOM [9], HART [10], MILLER [11], PONTER & LECKIE [12], CHABOCHE [13], KRIEG *et al.* [14], and ROBINSON [15]. Theoretical refinements have continued to occur throughout the 1980's in the models of CHABOCHE *et al.* [16], STOUFFER & BODNER [17], MARQUIS [18], VALANIS [19], WALKER [20], SCHMIDT & MILLER [21], CHABOCHE & ROUSSELIER [22], EISENBERG & YEN [23], BROWNING *et al.* [24], ESTRIN & MECKING [25], KREMPL *et al.* [26], LOWE & MILLER [27], ANAND & BROWN [28], HENSHALL *et al.* [29], McDOWELL & MOOSBRUGGER [30], NOUAILHAS [31], and FREED & WALKER [32]. Reviews on various aspects of viscoplasticity have been written by PERZYNA

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[33], WALKER [20], CHAN *et al.* [34], LEMAITRE & CHABOCHE [35], SWEARENGEN & HOLBROOK [36], EFTIS & JOLLES [37], MILLER [38], and CHABOCHE [39]. Although this listing is by no means complete, it does provide the reader with a representative bibliography of the work done in the field of viscoplasticity for initially isotropic materials.

This paper presents the development of a general, nonisothermal, viscoplastic theory from physical and thermodynamical considerations. The basic constitutive equations, evolution equations, and dissipation inequalities are derived for a class of materials whose virgin state is stress free and isotropic. Specific functional forms for the evolution equations are then proposed as they apply to a theory of viscoplasticity. Special attention is given to the intrinsic dissipation inequality. Here three fundamental internal state variables are considered. One accounts for kinematic or flow-induced anisotropic effects (*i.e.*, the tensor-valued back stress), while the other two account for isotropic effects (*i.e.*, the scalar-valued drag and yield strengths). Each is considered to evolve according to competitive phenomenological processes between strain hardening, strain-induced dynamic recovery, and time-induced thermal recovery, in accordance with current philosophies in the field (*cf.* review articles listed above). Of particular interest is the fact that each of the three basic internal variables can be composed as a sum of separately evolving constituents, which are the primary internal variables. Aspects of this feature have been used by CHABOCHE *et al.* [16,22,35,39], MILLER *et al.* [21,27,29,38], and others in their viscoplastic models.

We shall consider the thermo-mechanical behavior of viscoplastic materials; in particular, polycrystalline metals and their alloys. Each element of the material is assumed to be isotropic and to carry no stress in its initial virgin state. As the material deforms, however, anisotropies may become induced. Only kinematic anisotropy is considered in this paper. Small material displacements and rotations shall be considered to make up the deformation of the material. In a Cartesian reference configuration, the strain  $\epsilon_{ij}$  is taken to be composed of elastic  $\epsilon_{ij}^e$  (reversible - includes thermal strain) and inelastic or plastic  $\epsilon_{ij}^p$  (irreversible) parts such that

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p \quad (1)$$

and there is no inelastic strain in the stress-free virgin state.

## 2. THERMODYNAMICS

The changing internal structures of a material element are characterized by its state  $\{\epsilon_{ij}^e, a_\xi, T, \nabla_i T\}$ . The elastic strain  $\epsilon_{ij}^e$  is a measure for elastic changes in the internal structure. The internal state variables  $a_\xi$  ( $\xi=1, 2, \dots, n$ ) are measures for inelastic changes in the internal structure. The temperature  $T$  is a measure for the heat in a material element. And the thermal gradient  $\nabla_i T$  is a measure for heat fluxing from the element. An inelastic material response brought about by a change in the internal variables  $a_\xi$  affects the future response of the material. If the  $a_\xi$  are taken to be scalar valued, then nonintersecting subsets of the  $a_\xi$  may be taken to be components of higher-order, irreducible, even-rank tensors associated with a reference configuration [40].

For any quasistatic adiabatic process (which is also a *thermostatic* or reversible process), Gibbs equation is given by

$$dS = \frac{1}{T}(du - \sigma_{ij} d\epsilon_{ij}) \quad (2)$$

where  $S$  is the entropy,  $u$  is the internal energy,  $\sigma_{ij}$  is the applied stress, and the set  $\{S, \epsilon_{ij}\}$  defines the state. Entropy and temperature both exist in thermostatics, because  $dS$  is an exact differential and  $T > 0$  is its associated integrating factor [41]. It is from this relationship that one derives the theory of thermoelasticity. Here repeated Latin indicies are summed from 1 to 3 in the usual manner.

For any *thermodynamic* process, be it reversible or irreversible, the Gibbs equation of thermostatics generalizes to the inequality

$$\dot{S} \geq \frac{1}{T}(\dot{u} - \sigma_{ij} \dot{\epsilon}_{ij} + \frac{1}{T} q_i \nabla_i T) \quad (3)$$

where  $q_i$  is the heat flux, and the set  $\{S, \epsilon_{ij}, \nabla_i T\}$  defines the state. This relationship follows from the conservation of energy and the Clausius-Duhem inequality [42]. Here the existence of both entropy and temperature must be assumed to continue into the domain of irreversible processes, as rigorous justification for their existence continues to elude theoretical proof [43]. It is from this relationship that we derive our theory of viscoplasticity.

Thermostatics (equation (2)) addresses reversible processes where material response depends only on the initial and final states. Thermodynamics (equation (3)), on the other hand, addresses irreversible processes where material response not only depends on the initial and final states, but it also depends on the path traversed in state space.

Introducing the thermodynamic potential known as the Helmholtz free energy

$$\psi = u - ST \quad (4)$$

into equation (3), and using the decomposition of strain given in equation (1), one obtains the inequality

$$\sigma_{ij} \dot{\epsilon}_{ij}^P \geq \dot{\psi} + \dot{ST} - \sigma_{ij} \dot{\epsilon}_{ij}^e + \frac{1}{T} q_i \nabla_i T \quad (5)$$

From our experience with thermostatics, this relationship suggests that the set  $\{\epsilon_{ij}^e, \epsilon_{ij}^P, T, \nabla_i T\}$  should be the set of independent state variables. However, experimental evidence dictates that  $\epsilon_{ij}^P$  is not an admissible independent state variable (*cf.* ONAT & FARDSHISHEH [44]). Rather, one ought to introduce a set of internal state variables  $a_\xi$  ( $\xi=1, 2, \dots, n$ ) such that the set  $\{\epsilon_{ij}^e, a_\xi, T, \nabla_i T\}$  is taken as the set of independent state variables [45]. Therefore, considering the potential

$$\psi = \psi(\epsilon_{ij}^e, a_\xi, T, \nabla_i T) \quad (6)$$

equation (5) becomes

$$\sigma_{ij} \dot{\epsilon}_{ij}^P \geq \left[ \frac{\partial \psi}{\partial \epsilon_{ij}^e} - \sigma_{ij} \right] \dot{\epsilon}_{ij}^e + \left[ \frac{\partial \psi}{\partial T} + S \right] \dot{T} + \frac{\partial \psi}{\partial \nabla_i T} \frac{\partial \nabla_i T}{\partial t} + \sum_{\xi=1}^n \frac{\partial \psi}{\partial a_\xi} \dot{a}_\xi + \frac{1}{T} q_i \nabla_i T \quad (7)$$

But because this inequality must be valid for arbitrary changes in the observable variables  $\dot{\epsilon}_{ij}^e$ ,  $\dot{T}$ , and  $\partial \nabla_i T / \partial t$ , as the contributions from their associated terms are reversible, one obtains the following system of equations governing any thermodynamic process. They are [45]: the constitutive equations

$$\sigma_{ij} = \frac{\partial \psi}{\partial \epsilon_{ij}^e}, \quad A_\xi = \frac{\partial \psi}{\partial a_\xi}, \quad S = -\frac{\partial \psi}{\partial T}, \quad \frac{\partial \psi}{\partial \nabla_i T} = 0 \quad (8)$$

the dissipation inequality<sup>1</sup>

$$\sigma_{ij} \dot{\epsilon}_{ij}^p - \sum_{\xi=1}^n A_\xi \dot{a}_\xi - \frac{1}{T} q_i \nabla_i T \geq 0 \quad (9)$$

and the evolution equations

$$\dot{\epsilon}_{ij}^p = \dot{\epsilon}_{ij}^p(\epsilon_{kl}^e, a_\gamma T, \nabla_k T), \quad \dot{a}_\xi = \dot{a}_\xi(\epsilon_{kl}^e, a_\gamma T, \nabla_k T), \quad q_i = q_i(\epsilon_{kl}^e, a_\gamma T, \nabla_k T) \quad (10)$$

where  $A_\xi$  ( $\xi=1, 2, \dots, n$ ) are the thermodynamic forces or affinities conjugate to the thermodynamic displacements  $a_\xi$  or fluxes  $\dot{a}_\xi$ . Notice that the Helmholtz free energy is independent of the thermal gradient [42,45].

Our past observations of material response allow us to consider  $q_i$  as being independent of  $\epsilon_{ij}^e$ ,  $a_\xi$  and  $T$ , explicitly, and to also consider both  $\dot{\epsilon}_{ij}^p$  and  $\dot{a}_\xi$  as being independent of  $\nabla_i T$ , explicitly. Hence, the total dissipation inequality given in equation (9) can be separated into the intrinsic dissipation inequality given by

$$\sigma_{ij} \dot{\epsilon}_{ij}^p - \sum_{\xi=1}^n A_\xi \dot{a}_\xi \geq 0 \quad (11)$$

and the thermal dissipation inequality given by

$$\frac{1}{T} q_i \nabla_i T \leq 0 \quad (12)$$

Usually, the evolution of heat flux is represented by Fourier's equation

$$q_i = -k_{ij} \nabla_j T \quad (13)$$

where the symmetric tensor for thermal conductivity  $k_{ij}$  must be positive definite in order to satisfy the thermal dissipation inequality.

### 3. CONSTITUTIVE EQUATIONS

The constitutive equations derived from thermodynamic considerations in the previous section are very general. However, by choosing a particular form for the thermodynamic potential  $\psi$  based upon our past experiences, these general relationships become specific and useful.

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<sup>1</sup> Some authors (e.g., COLEMAN & GURTIN [42]) would divide this inequality by temperature and set it equal to a non-negative term they call the rate of entropy production, but this is an unnecessary definition in our development.

Most existing viscoplastic models (without damage) belong to the class of materials characterized by the thermodynamic potential

$$\Psi = \Psi(\epsilon_{ij}^e, a_\xi, T) = \Psi^e(\epsilon_{ij}^e, T) + \Psi^p(a_\xi; T) \quad (14)$$

such that

$$\begin{aligned} \Psi^e &= \frac{1}{2} \epsilon_{ij}^e D_{ijkl} \epsilon_{kl}^e - \epsilon_{ij}^e D_{ijkl} \alpha_{kl} \Delta T - C^e T \ln\left(\frac{T}{T_0}\right) + C^e \Delta T \\ \Psi^p &= \frac{1}{2} \sum_{\xi, \gamma=1}^n a_\xi H_{\xi\gamma} a_\gamma \geq 0 \end{aligned} \quad (15)$$

where  $T_0$  is the initial value of temperature with  $\Delta T = T - T_0$ ,  $C^e = T \partial S(\epsilon_{ij}^e, T)/\partial T$  is the heat capacity at constant elastic strain,  $\alpha_{ij}$  are the thermal expansion coefficients,  $D_{ijkl}$  are the elastic moduli, and  $H_{\xi\gamma}$  is the symmetric matrix of hardening coefficients (which is taken to be positive definite). These material constants typically vary with temperature to some degree. It is in this sense that the plastic potential  $\Psi^p$  depends on temperature. This is an implicit dependence rather than an explicit one, however, as we imply by the notation  $(;T)$ . The fact that the plastic potential  $\Psi^p$  is taken as a positive-definite quadratic form is a stronger requirement than is actually necessary, since only the property of convexity is required of this potential.

By substituting the thermodynamic potentials of equation (14) into the constitutive equations (8), and using the decomposition of strain given in equation (1), one obtains the following system of equations which govern any process for the class of viscoplastic materials considered. They are: the constitutive equations

$$\begin{aligned} \epsilon_{ij} &= D_{ijkl}^{-1} \sigma_{kl} + \alpha_{ij} \Delta T + \dot{\epsilon}_{ij}^p \\ S &= \alpha_{ij} \sigma_{ij} + C^\sigma \ln\left(\frac{T}{T_0}\right) \\ A_\xi &= \sum_{\gamma=1}^n H_{\xi\gamma} a_\gamma \end{aligned} \quad (16)$$

the dissipation inequalities

$$\sigma_{ij} \dot{\epsilon}_{ij}^p - \sum_{\xi=1}^n A_\xi \dot{a}_\xi \geq 0 \quad , \quad \frac{1}{T} q_i \nabla_i T \leq 0 \quad (17)$$

and the evolution equations

$$\dot{\epsilon}_{ij}^p = \dot{\epsilon}_{ij}^p(\sigma_{kl}, A_\gamma, T) \quad , \quad \dot{a}_\xi = \dot{a}_\xi(\sigma_{kl}, A_\gamma, T) \quad , \quad q_i = -k_{ij} \nabla_j T \quad (18)$$

where  $C^\sigma = T \partial S(\sigma_{ij}, T)/\partial T$ , and therefore  $C^\sigma = C^e + \alpha_{ij} D_{ijkl} \alpha_{kl}$  in the limit as  $\Delta T \rightarrow 0$ , which denotes the heat capacity at constant stress or pressure. Here it is assumed that  $S = 0$  when  $\Delta T = \sigma_{ij} = 0$ . These governing equations reduce to those of thermoelasticity [46] in the absence of inelastic deformation. The thermodynamic forces replace the thermodynamic displacements as the arguments of the evolution equations without loss in generality because the constitutive equations for internal state are linear. This fundamental set of

equations forms the basis upon which we develop our viscoplastic theory in the next section.

Since this paper is directed towards materials that are isotropic in their initial virgin state, one requires that

$$D_{ijkl}^{-1} = \frac{1+\nu}{E} \delta_{ik} \delta_{jl} - \frac{\nu}{E} \delta_{ij} \delta_{kl} , \quad \alpha_{ij} = \alpha \delta_{ij} , \quad k_{ij} = k \delta_{ij}$$

$$A_\xi = H_{\xi\gamma} a_\gamma \rightarrow A = ha \quad \text{for all scalar state variables } a_\gamma \rightarrow a \quad (19)$$

$$A_\xi = \sum_{\gamma=1}^{n+6} H_{\xi\gamma} a_\gamma \rightarrow A_{ij} = H \delta_{ik} \delta_{jl} a_{kl} = Ha_{ij} \quad \text{for all tensor state variables } a_\gamma \rightarrow a_{ij} = a_{ji} , a_{ii} = 0$$

Furthermore, isotropy also requires that the inelastic strain, and all internal state variables that are associated with second-rank tensors, to be zero-valued in the initial virgin state, *i.e.*,

$$\varepsilon_{ij}^p(t=0) = 0 , \quad a_{ij}(t=0) = 0 \quad (20)$$

Although internal state variables associated with fourth and higher even-rank tensors are admissible, they are not required in this work. Here  $E$  is the elastic modulus,  $\nu$  is the Poisson ratio,  $\alpha$  is the coefficient of thermal expansion, and  $k$  is the thermal conductivity, all of which vary with temperature, typically. The hardening coefficients for the internal variables  $h$  and  $H$  may vary with temperature, too. The Kronecker delta  $\delta_{ij}$  is defined as 1 whenever  $i=j$ , otherwise it is 0 valued. Since all internal state variables that are second-rank tensors must be irreducible [40], they must either be symmetric and traceless (*i.e.*, deviatoric) or antisymmetric, but only deviatoric tensors shall be considered here.

#### 4. VISCOPLASTIC THEORY

In the preceeding section, structure was given to the constitutive equations. In this section, structure is given to the evolution equations resulting in a general theory of viscoplasticity. Once again we use our past experiences to guide us, this time for the purpose of proposing functional forms for the evolution equations. One does not have complete freedom, however, in choosing these forms. They must, by necessity, satisfy the thermodynamic constraint of intrinsic dissipation (see equation (11)); just as Fourier's equation (13) must satisfy the thermodynamic constraint of thermal dissipation (which constrains the thermal conductivity  $k_{ij}$  to be positive definite).

The total dissipation inequality of equation (9) has the form

$$\sum_{\xi=1}^m X_\xi \dot{Y}_\xi \geq 0 \quad (21)$$

where the  $X_\xi$  represent the thermodynamic affinities  $\{\sigma_{ij}, -A_\gamma, -\nabla T_i/T\}$  and the  $\dot{Y}_\xi$  represent their associated thermodynamic fluxes  $\{\dot{\varepsilon}_{ij}^p, \dot{a}_\gamma, \dot{q}_i\}$  characterized through the evolution equations (18). As the simplest case, the linear evolution equations of ONSAGER [47]

$$\dot{Y}_\xi = \sum_{\gamma=1}^m L_{\xi\gamma} X_\gamma \quad (22)$$

apply to some neighborhood - be it large or small - of every equilibrium state, and as such, the total dissipation of the material element is a minimum in that neighborhood [48]. Here the symmetric matrix of coefficients  $L_{\xi\gamma}$  must be positive definite in order to satisfy the thermodynamic dissipation inequality given in equation (21). Fourier's evolution equation (13) for heat flux satisfies such a linear relationship; however, a linear relationship for the evolution of inelastic strain only exists for Coble and Nabarro-Herring creep in metallic materials (cf. ASHBY [49]). Neither of these creep mechanisms are of interest to viscoplasticists, because the stress levels associated with Coble and Nabarro-Herring creep are too small for the viscoplasticists' intended applications. For those applications of interest to the viscoplastic modeler, the inelastic strain rate is a highly nonlinear function of stress. Hence, *the thermodynamics of Onsager does not apply to viscoplasticity*. We are therefore left to consider a more phenomenological approach.

BRIDGMAN's [50] experimental results demonstrate that the evolution of inelastic strain in metals is, to a good approximation, insensitive to moderate levels of hydrostatic pressure. Consequently, the evolution equations (18) for inelastic strain and the internal state variables take on the more restricted form

$$\dot{\epsilon}_{ij}^p = \dot{\epsilon}_{ij}^p(S_{kl}, A_\gamma, T) , \quad \dot{a}_\xi = \dot{a}_\xi(S_{kl}, A_\gamma, T) \quad (23)$$

where  $S_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\delta_{ij}$  is the deviatoric stress (not to be confused with the entropy  $S$ ).

The stress dependence of inelastic flow is strongly influenced by the material's anisotropy, which may be either inherent, or flow-induced, or both. In this paper only flow-induced anisotropy is considered, and it is introduced after the manner of PRAGER [51]. This is done through an internal variable  $B_{ij}$  called the back or internal stress, which is a symmetric and traceless tensor (*i.e.*, deviatoric), and therefore irreducible in accordance with the theoretical results from group theory of GEARY & ONAT [40]. To assure an initial condition that is isotropic,  $B_{ij}(t=0) = 0$ .

von MISES [52] introduced the concept of a plastic potential  $F$  in the flow equation for inelastic strain, which we write as

$$\dot{\epsilon}_{ij}^p = |\dot{\epsilon}^p| \frac{\partial F}{\partial \sigma_{ij}} \quad (24)$$

where  $\partial F / \partial \sigma_{ij}$  defines the direction (in unit length) of inelastic straining, with  $|\dot{\epsilon}^p|$  providing its magnitude. The existence of this potential followed naturally from von MISES' [53] prior definition of yield, which is equivalent to introducing a second-invariant (or quadratic) norm that we write as

$$|\dot{\epsilon}^p| = \sqrt{2/3 \dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p} , \quad |J| = \sqrt{3/2 J_{ij} J_{ij}} \quad (25)$$

where the tensor  $J_{ij} \in \{S_{ij}, B_{ij}, S_{ij} - B_{ij}\}$ . More recently, RICE [54] demonstrated the physical existence of such a potential function based upon the mechanics and thermodynamics of dislocation slip. Most viscoplastic models take  $F = |S - B|$  such that equation (24) becomes

$$\dot{\epsilon}_{ij}^p = \frac{3}{2} |\dot{\epsilon}^p| \frac{S_{ij} - B_{ij}}{|S - B|} \quad (26)$$

which is compatible with the kinematic constructs proposed by PRAGER [51] in his plasticity model. The choice of this plastic potential provides a reasonable approximation to the actual shapes of observed yield and flow surfaces [55]. The norms given in equation (25) are scaled for tension. They could have just as easily been scaled for shear by replacing the coefficients that appear under the radical signs with a  $1/2$  for  $S_{ij}B_{ij}$ , and  $S_{ij}-B_{ij}$ , and a  $2$  for  $\dot{\epsilon}_{ij}^P$ .

ZENER & HOLLOWOM's [56] experimental results demonstrate that the functional dependence for the magnitude of inelastic strain rate  $|\dot{\epsilon}^P|$  can, to a good approximation, be decomposed into the product of functions

$$|\dot{\epsilon}^P(S_{ij}A_\beta, T)| = \theta(T) Z(S_{ij}A_\beta; T) \geq 0 \quad (27)$$

where  $\theta > 0$  is the thermal diffusivity, and  $Z \geq 0$  shall be called the Zener parameter. The thermal diffusivity is often represented as an Arrhenius function of temperature, which is valid over a significant but specified temperature range (*cf.* MILLER [11]). The functional dependence of the Zener parameter is of particular interest to us in this paper. This parameter is a temperature normalized function for the magnitude of inelastic strain rate that, at most, has only an implicit temperature dependence.

In addition to the kinematic variable, or back stress  $B_{ij}$ , there are two isotropic variables introduced into the general structure of our theory; they are the yield strength  $Y \geq 0$  and the drag strength  $D > 0$  (not to be confused with the elastic moduli  $D_{ijkl}$ ). These three internal variables interact with the deviatoric stress  $S_{ij}$  in such a way that the Zener parameter of equation (27) is considered to have the following functional dependence

$$Z = Z\left[\frac{<|S-B|-Y>}{D}\right] = Z\left[\frac{<\sigma^v>}{D}\right] = \frac{|\dot{\epsilon}^P|}{\theta} \geq 0 \quad (28)$$

in accordance with existing viscoplastic models. Here  $\sigma^v = |S-B|-Y$  is the viscous stress governing the inelastic material response, with  $Z(0) = 0$ . The Macauley bracket  $<\sigma^v>$  has either a value of  $0$  whenever  $|S-B| \leq Y$  (defining the elastic or reversible domain) or a value of  $\sigma^v$  whenever  $|S-B| \geq Y$  (defining the inelastic or irreversible domain), with  $|S-B| = Y$  establishing the yield surface. Viscoplasticity (a rate dependent theory) admits states within the yield surface, on the yield surface, and outside the yield surface. Plasticity (a rate independent theory), on the other hand, only admits states within and on the yield surface; it does not admit states outside the yield surface. The expression for the Zener parameter given above is very general, and includes three important special cases: *i*) viscoplastic theories without a yield surface (*i.e.*,  $Y=0$ ) as used by MILLER [11], WALKER [20], KREMPI *et al.* [26], and others; *ii*) viscoplastic theories with no evolving drag strength (*i.e.*,  $D = \text{constant}$ ) as used by CHABOCHE [13], MARQUIS [18], and others; and *iii*) viscoplastic theories where the yield and drag strengths are proportional (*i.e.*,  $Y \propto D$ ) as used by PERZYNA [7], ROBINSON [15], and NOUAILHAS [31].

Phenomenologically, each internal variable  $A \in \{B_{ij}, D, Y\}$  is taken to competitively evolve through a process of the type (*cf.* CHABOCHE [13,39], WALKER [20], and MILLER *et al.* [27,38])

$$\dot{A} = \text{hardening} - \text{dynamic recovery} - \text{static recovery} \quad (29)$$

where the hardening term accounts for strengthening mechanisms, and the recovery terms account for softening mechanisms. The hardening and dynamic recovery terms both evolve with inelasticity; they are strain induced. The static recovery term evolves thermally; it is time induced. What functional form each particular term takes on varies with the type of internal variable, as shown below.

At this time we list the basic structure for a thermodynamically admissible theory of viscoplasticity. A detailed discussion of the evolution equations follows in the next section. Our general theory of viscoplasticity for initially isotropic materials is characterized by the *constitutive equations*

$$\begin{aligned} S_{ij} &= 2\mu(e_{ij} - \epsilon_{ij}^p) , \quad \sigma_{ii} = 3K(\epsilon_{ii} - \alpha \Delta T \delta_{ii}) \\ B_{ij} &= \sum_{\xi=1}^u B_{ij}^{\xi} = \frac{2}{3} \sum_{\xi=1}^u H_{\xi} \beta_{ij}^{\xi} \\ D - D_0 &= \sum_{\zeta=1}^v D_{\zeta} = \sum_{\zeta=1}^v h_{\zeta} \delta_{\zeta} \\ Y - Y_0 &= \sum_{\gamma=1}^w Y_{\gamma} = \sum_{\gamma=1}^w h_{\gamma}' y_{\gamma} \end{aligned} \quad (30)$$

by the *evolution equations*

$$\begin{aligned} \dot{\epsilon}_{ij}^p &= \theta Z n_{ij} = |\dot{\epsilon}^p| n_{ij} \\ \dot{\beta}_{ij}^{\xi} &= \dot{\epsilon}_{ij}^p - \frac{|B^{\xi}|}{L_{\xi}} |\dot{\epsilon}^p| d_{ij}^{\xi} - \theta R_{\xi} b_{ij}^{\xi} \\ \dot{\delta}_{\zeta} &= c \left[ \left( 1 - \frac{D_{\zeta}}{l_{\zeta}} \right) |\dot{\epsilon}^p| - \theta r_{\zeta} \right] \\ \dot{y}_{\gamma} &= \left[ 1 - \frac{Y_{\gamma}}{l_{\gamma}'} \right] |\dot{\epsilon}^p| - \theta r_{\gamma}' \end{aligned} \quad (31)$$

with directions (unit or projection) defined as

$$\begin{aligned} n_{ij} &= \frac{3}{2} \frac{S_{ij} - B_{ij}}{|S - B|} , \quad b_{ij}^{\xi} = \frac{3}{2} \frac{B_{ij}^{\xi}}{|B^{\xi}|} , \quad n_{ij}^{\xi} = \frac{3}{2} \frac{S_{ij} - B_{ij}^{\xi}}{|S - B^{\xi}|} \\ d_{ij}^{\xi} &= (1 - p_{\xi}) b_{ij}^{\xi} + p_{\xi} \frac{n_{kl}^{\xi} B_{kl}^{\xi}}{|B^{\xi}|} n_{ij}^{\xi} \end{aligned} \quad (32)$$

and therefore, by the *intrinsic dissipation inequality*

$$\left\{ \sigma^v + Y_0 - c(D - D_0) + \sum_{\xi=1}^u |B_\xi| \left[ \frac{|B_\xi|}{L_\xi} \left( 1 - p_\xi \left[ 1 - \left( \frac{n_{ij}\xi B_{ij}}{|B_\xi|} \right)^2 \right] \right) + \frac{R_\xi}{Z} \right] + \right. \\ \left. + c \sum_{\zeta=1}^v D_\zeta \left[ \frac{D_\zeta}{l_\zeta} + \frac{r_\zeta}{Z} \right] + \sum_{\gamma=1}^w Y_\gamma \left[ \frac{Y_\gamma}{l'_\gamma} + \frac{r'_\gamma}{Z} \right] \right\} |\dot{\epsilon}^P| \geq 0 \quad (33)$$

Here  $\mu = E/2(1+v)$  is the elastic shear modulus;  $K = E/3(1-2v)$  is the elastic bulk modulus;  $e_{ij} = \epsilon_{ij} - (\epsilon_{kk}/3)\delta_{ij}$  is the deviatoric strain;  $\beta_{ij}$ ,  $\delta$  and  $\dot{\gamma}$  are the thermodynamic fluxes conjugate to the thermodynamic forces  $B_{ij}$ ,  $D - D_0$ , and  $Y - Y_0$ ; and  $H$ ,  $h$ , and  $h'$  are their associated hardening coefficients. The material constants of the theory are:  $D_0 > 0$  is the initial value of drag strength,  $Y_0 \geq 0$  is the initial value of yield strength,  $p_\xi \in [0,1]$  is a nonproportionality parameter associated with the  $\xi^{th}$  constituent of back stress, and  $c \in (0,1]$  is a parameter that can be arbitrarily set to ensure satisfaction of the intrinsic dissipation inequality, equation (33). In the virgin or annealed state,  $B_{ij} = 0$ ,  $D = D_0$ , and  $Y = Y_0$ ; therefore,  $B_{ij}^\xi = 0$ ,  $D_\zeta = 0$ , and  $Y_\gamma = 0$  in this state. The material functions of the theory are:  $\theta > 0$  is the thermal difusivity;  $Z = |\dot{\epsilon}^P|/\theta \geq 0$  is the Zener parameter;  $L > 0$ ,  $l > 0$ , and  $l' > 0$  are the limiting states for the internal variables, as associated with dynamic recovery; and  $R \geq 0$ ,  $r \geq 0$ , and  $r' \geq 0$  are the static recovery functions. The limit and static-recovery functions are, in general, functions of state as established by the set  $\{S_{ij}, B_{ij}, D, Y, T\}$ , while the functional dependences of thermal diffusivity and Zener's parameter are specified in equations (27 and 28). These constitutive, evolution, and dissipation equations are a special case of the thermodynamic relationships given in equations (16-18).

The kind and the number of independent state variables chosen, and the specific forms chosen for their associated material functions, vary from model to model. A particular model may be simple and employ only a couple facets of the theory, or it may be complex and employ many facets of the theory. This discretion is left to the individual modeler who best understands his own needs.

In addition to equations (30-33) which characterize the deformation properties of the material, there are the constitutive equation for entropy, the evolution equation for heat flux, and the thermal dissipation inequality given in equations (16-18) which characterize the thermal properties of the material.

## 5. DISCUSSION OF THE THEORY

An arbitrary number (*i.e.*,  $u$ ) of constituent back stresses  $B_{ij}^\xi = 2H_\xi\beta_{ij}^\xi/3$  can be introduced into a viscoplastic model. It has been demonstrated by CHABOCHE *et al.* [16,22,35,39], WALKER [20], MILLER *et al.* [27,29,38], and McDOWELL & MOOSBRUGGER [30] that two back stresses (sometimes referred to as short-range and long-range back stresses) are typically necessary for proper material characterization. They are thermodynamically admissible within this theoretical framework provided that *i*) the back stress is given by the entire sum of its constituents  $B_{ij} = \sum_{\xi=1}^u B_{ij}^\xi$ , and that *ii*) each constituent evolves in a manner compatible with equation

(31). The back stress is therefore a sum of shape or weight functions, each being its own, generalized, state-variable coordinate.

The implementation of the phenomenology of competing hardening and recovery mechanisms into an evolution equation for the back stress is complicated by the fact that the back stress is a tensorial quantity, and therefore has directional characteristics. Here the back stress is considered to harden linearly in the same direction that the inelastic strain evolves in, as advocated by PRAGER [51]. Static recovery is taken to occur in a direction that opposes itself, in accordance with the Bailey-Orowan model of PONTER & LECKIE [12]. And for the dynamic recovery of back stress, we adopt the concept of BURLET & CAILLETAUD [57], as it presents a natural generalization to the popular model of ARMSTRONG & FREDERICK [8]. Figure 1 presents a schematic for the directional characteristics of this phenomenology. It is simplified here for the case of one constituent of back stress for illustrative purposes. The Armstrong-Frederick model considers dynamic recovery to occur in a direction that opposes itself (like static recovery), resulting in a formulation that is equivalent to a two-surface theory of plasticity with Mroz hardening, as shown in reference 18. The Burlet-Cailletaud model is equivalent to the Armstrong-Frederick model for proportional loading histories, but it has the capability for predicting a different response under nonproportional conditions, depending upon the value of the nonproportionality parameter  $p_\xi$  found in equation (32). The one extreme of  $p_\xi = 0$  gives dynamic recovery as advocated by Armstrong and Frederick. The other extreme of  $p_\xi = 1$  causes dynamic recovery for the  $\xi^k$  constituent of back stress to occur in a direction  $-n_{ij}^\xi$ . This direction does not oppose the direction of back stress; rather, it opposes that of inelastic strain rate (for the case of one constituent in back stress, as in the Burlet-Cailletaud model), which generalizes Prager's, linear, kinematic, hardening rule into a nonlinear, kinematic, hardening rule. These two extremes can predict vastly different nonproportional responses. For example, tension-torsion ratchetting behavior need not shakedown for the case of  $p_\xi = 0$ , while shakedown occurs almost immediately for the case of  $p_\xi = 1$  [57]. Reality is likely to be somewhere between these two extremes, but probably closer to Armstrong-Frederick (*i.e.*  $p_\xi = 0$ ) than nonlinear Prager (*i.e.*  $p_\xi = 1$ ) [32]. The coefficient  $n_k^\xi B_k^\xi / |B^\xi|$  to the unit normal  $n_{ij}^\xi$  appearing in equation (32) is necessary for two reasons. First, it is required to obtain an equivalent predicted response under proportional loading conditions for all values of  $p_\xi$ . And second, it guarantees a positive-valued contribution to the intrinsic dissipation inequality (*i.e.*,  $(n_{ij}^\xi B_{ij}^\xi)^2$  since  $n_{ij}^\xi B_{ij}^\xi$  need not be - and often is not - positive valued).

The basic form of the evolutionary equation for back stress, as proposed by ARMSTRONG & FREDERICK [8], has been used by CHABOCHE *et al.* [13,16,22], WALKER [20], and many others in their viscoplastic models. In the absence of static recovery, this relationship introduces an evanescent strain memory effect (evanescent along the inelastic strain path) caused by a strain-induced dynamic recovery phenomenon which competes against the mechanism of linear strain hardening. This results in the back stress  $B_{ij}$  evolving asymptotically to a limit state  $L$  associated with kinematic saturation. Considering just one constituent for illustration, one can combine the constitutive and evolution equations (30 and 31) for back stress into a form of the type proposed by Armstrong and Frederick; it is

$$\dot{B}_{ij} = \frac{2}{3} \lambda \left[ \dot{\epsilon}_{ij}^P - \phi |\dot{\epsilon}^P| b_{ij} \right] \quad (34)$$

under isothermal conditions. (Nonisothermal conditions would bring about an influence due to the temperature dependence of the hardening coefficient  $H$  - cf. CHABOCHE [13,39] and WALKER [20].) Convergence is exponential whenever  $\lambda$  and  $\phi$  are constants [8], but in our general theory

$$\lambda = H \left[ 1 - p \frac{n_{ij} B_{ij}}{L} \right] \quad \text{and} \quad \phi = \frac{(1-p) \frac{|B|}{L} + \frac{R}{Z}}{1 - p \frac{n_{ij} B_{ij}}{L}} \quad (35)$$

Hence, the model of BURLET & CAILLETAUD [57] effectively changes the concept of linear strain hardening into one of nonlinear strain hardening for the evolution of back stress. A graphic illustration of this evanescent concept is given in figure 2. Here the limiting state of back stress defines a hypersurface  $\phi = 1$  in state space where  $|B| = B_{sat}$ . Whenever this bound is reached, a perfectly plastic or steady-state creep response is attained. In the absence of static recovery,  $B_{sat} = L$  where  $L$  is the rate-independent (or plastic) limiting state of back stress; that is,  $L$  is the upper bound of  $B_{sat}$ . The value of this limit state is known to vary as the strength of the material varies [18,20], i.e.,  $L(D, Y, T)$ . The presence of static recovery reduces the limiting state of back stress making it rate-dependent, such that  $0 \leq B_{sat} \leq L$ . In the absence of dynamic recovery, the evolution equation (31) is in accord with the BAILEY-OROWAN [58,59] hypothesis, that creep deformation evolves through a competitive process between the mechanisms of strain hardening and time-induced static (or thermal) recovery. At kinematic saturation, the inelastic strain rate  $\dot{\epsilon}_{ij}^P$  becomes coaxial with both the deviatoric stress  $S_{ij}$  and the back stress  $B_{ij}$  (in agreement with experiment [60,61]), and the nested set of flow surfaces  $\{|S - B| = \text{constant}\}$  becomes stationary until unloading occurs. Otherwise, this set of flow surfaces can translate freely within the limiting hypersurface, as governed by the flow and evolution equations.

Viscoplastic models use either inelastic strain (i.e.  $|\dot{\epsilon}^P|$ ) or inelastic work (i.e.  $S_{ij} \dot{\epsilon}_{ij}^P$  or  $(S_{ij} - B_{ij}) \dot{\epsilon}_{ij}^P$ ) to drive the evolution of the isotropic variables  $D$  and  $Y$ . Note that  $(S_{ij} - B_{ij}) \dot{\epsilon}_{ij}^P$  is the proper expression to use for inelastic work when a back stress is present, since in this case  $S_{ij} \dot{\epsilon}_{ij}^P$  is not always positive valued. A set of critical experiments has been performed by KREML [62] to determine which of these forms for the evolution of internal state is most representative of material response. He determined that inelastic strain provides the best characterization of strain hardening behavior.

A hardening/recovery format for the evolution of drag strength  $D$  has been used by BODNER *et al.* [9,17], MILLER *et al.* [11,21,27,29], WALKER [20], and others in their viscoplastic models. Bodner uses inelastic work to drive its evolution, while the others, for the most part, use inelastic strain for this purpose. Since a parameter  $c \in (0,1]$  can be arbitrarily introduced into an evolution equation for drag strength (e.g. see equation (31)), either hardening mechanism (i.e. inelastic strain or inelastic work) can be made to satisfy the intrinsic dissipation inequality of equation (11). Actually, the parameter  $c$  is transparent to the viscoplastic modeler, since he has the flexibility to arbitrarily adjust the hardening coefficient  $h$  found in the constitutive equation (30). There is only one driver for the evolution of drag strength that satisfies intrinsic dissipation independent of a parameter  $c$ ; it is  $(S_{ij} - B_{ij}) \dot{\epsilon}_{ij}^P / (D - D_0)$  as introduced by FREED & WALKER [32]. Multiple drag strengths are admissible

within the constructs of this theory, *i.e.*  $D = D_0 + \sum_{\zeta=1}^v D_\zeta$ . They have been advocated in the models of MILLER *et al.* [21,27,29,38] to complement the short- and long-range back stresses, and to account for solute strengthening effects. Except for the fact that the evolutionary equation for drag strength given in equations (30 and 31) is a scalar equation instead of a tensorial one, it has the same mathematical features as the evolutionary equation for back stress.

The evolution equation for yield strength  $Y$  given in equations (30 and 31) is used in the models of CHABOCHE [13,16,22,39]. Our theory also permits the yield strength to be given as a sum of constituents  $Y = Y_0 + \sum_{\gamma=1}^w Y_\gamma$ ; however, there seems to be no physical significance in permitting  $w$  to exceed 1. The evolution of yield strength has the same mathematical features as the evolution of drag strength, except that a parameter like  $c$  is not required to satisfy intrinsic dissipation.

There are two types of viscoplastic models typically found in the literature; those with a yield surface, and those without one. Each is a special case of our theory. Each has a different energy-storage/heat-dissipation diagram, as illustrated in figure 3. If one considers a material whose internal state is fully recovered or annealed, and if one deforms this material in monotonic tension in the absence of thermal recovery, then for those models with no evolving drag strength, *i.e.*,  $D = D_0$ , the intrinsic dissipation inequality (equation (33)) becomes

$$\left[ \sigma^\nu + Y_0 + \frac{B^2}{L} + \frac{(Y - Y_0)^2}{l'} \right] \dot{\epsilon}^p \geq 0 \quad (36)$$

while for those models with no yield surface, *i.e.*,  $Y = Y_0 = 0$ , it becomes

$$\left[ \sigma^\nu - c(D - D_0) + \frac{B^2}{L} + c \frac{(D - D_0)^2}{l} \right] \dot{\epsilon}^p \geq 0 \quad (37)$$

Here only one constituent for each internal variable is considered for simplicity. These are the equations depicted in figures 3a and 3b, respectively. There the isotropic and kinematic variables are shown to saturate at the same level of accumulated inelastic strain, which is not an accurate representation of our observations, but it is a convenient one for illustrative purposes. These models predict energy-storage/heat-dissipation ratios that differ between themselves, but probably not to a large extent. As a consequence of effecting the intrinsic dissipation, the parameter  $c$  in the drag strength model influences the amount of inelastic work done on a body that is stored internally versus that which is dissipated away as heat. At saturation, both models predict that all the inelastic work done  $S \dot{\epsilon}^p$  is dissipated away as heat, in agreement with experiment [63].

## 6. SUMMARY

Constitutive, evolution, and dissipation equations governing the response of a general class of materials are derived from physical and thermodynamical considerations. These equations apply to materials that are initially isotropic, and whose strain is additively decomposed into elastic (reversible) and inelastic (irreversible) components. A general structure for a theory of viscoplasticity is then proposed. A second invariant formulation with three fundamental internal state variables is considered. One internal variable (the tensor-valued back stress) accounts for kinematic or flow-induced anisotropic effects. The other two variables (the scalar-valued drag and yield strengths) account for isotropic effects. Each can be composed as a sum of independently evolving constituents. All the internal variables are assumed to evolve phenomenologically through a competitive process of strain hardening, strain-induced dynamic recovery, and time-induced static recovery. The thermodynamic constraint of intrinsic dissipation requires attention only for the hardening term of drag strength. By introducing an arbitrary parameter  $c$  into the evolution equation for drag strength, the constraint of intrinsic dissipation can be satisfied.

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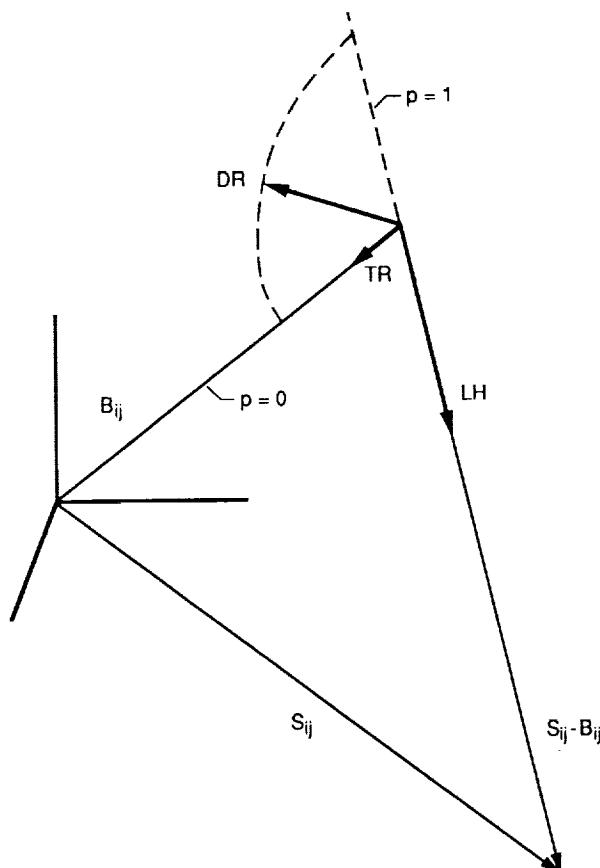


Figure 1. - Schematic of directions for linear hardening  $LH$ , dynamic recovery  $DR$ , and thermal recovery  $TR$  of back stress.

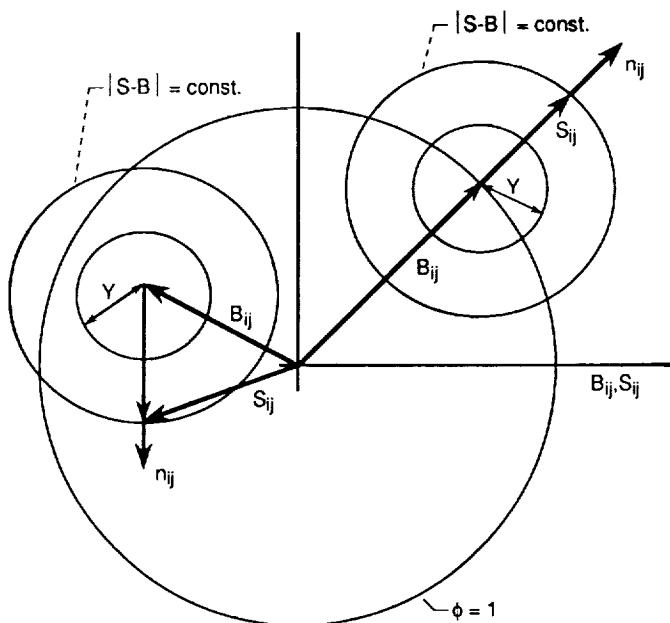
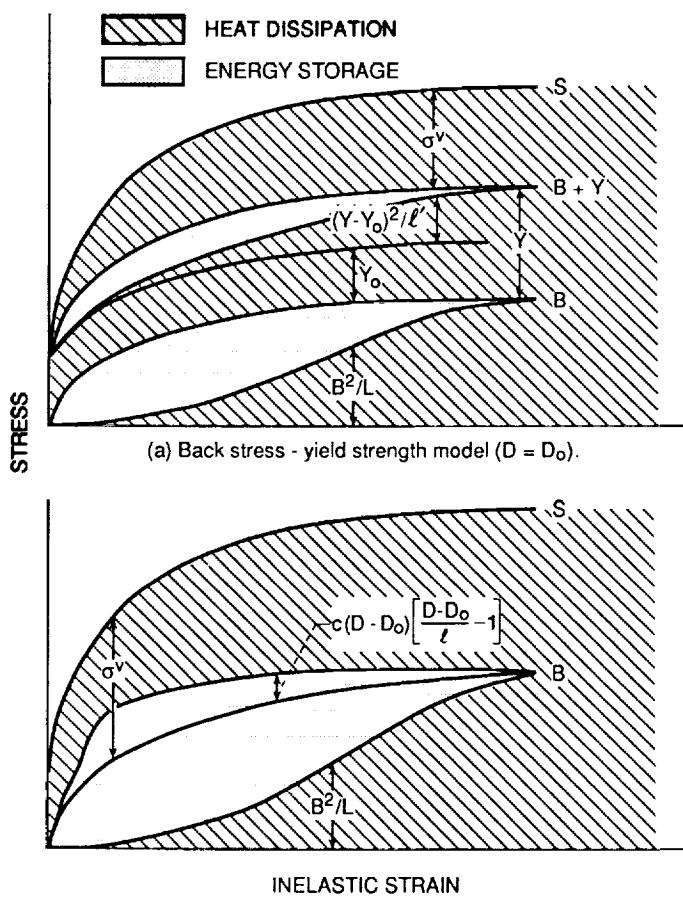


Figure 2. - State space representation of the evanescent character of back stress.



(b) Back stress - drag strength model ( $Y = 0$ ).  
Figure 3. - Energy storage and heat dissipation due to inelastic deformation.



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16. Abstract A thermodynamic foundation using the concept of internal state variables is given for a general theory of viscoplasticity, as it applies to initially isotropic materials. Three fundamental internal state variables are admitted; they are: a tensor-valued back stress for kinematic effects, and the scalar-valued drag and yield strengths for isotropic effects. All three are considered to phenomenologically evolve according to competitive processes between strain hardening, strain-induced dynamic recovery, and time-induced static recovery. Within this phenomenological framework, a thermodynamically admissible set of evolution equations is put forth. This theory allows each of the three fundamental internal variables to be composed as a sum of independently evolving constituents.			
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